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Generalized Noether Theorem and Poincaré Invariant for Nonconservative Nonholonomic Systems

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We generalize Noether's theorem and the Poincaré invariant to conservative and nonconservative systems with nonlinear nonholonomic constraints. The conservation laws of such systems are illustrated.

1. INTRODUCTION

The connection between symmetry and conservation laws is usually referred to as Noether's theorem. In most cases, Noether's theorem is valid only for systems which are completely describable by means of a Lagrangian whose state functions or generalized coordinates are independent variables. However, there are many constrained systems whose motion may be described in term of nonindependent state functions or generalized coordinates. In addition, some nonconservative systems cannot be completely described by means of a Lagrangian. Then the inclusion of constrained nonconservative systems into the theory is necessary. Linear constrained systems were discussed by Li (1981, 1984) and Bahar and Kwatny (1987). A system whose motion is described by quasicoordinates was discussed by Djukic (1974). The generalization of Noether's theorem for nonconservative systems was given by Vujanovic (1975, 1978) and Vujanovic *et al.*, (1986) and other discussions were given by Djukic and Strauss (1980), Djukic and Vujanovic (1975), Djukic and Sutela (1984), and Sarlet and Bahar (1981). Here we give a brief discussion of conservative and nonconservative system with nonlinear nonholonomic constraints, where the existence of a corresponding variational principle is not imposed. Considering the transformation properties of such a system, we give a generalization of Noether's

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theorem. It is a slight extension of the previous results for the nonconservative nonlinear nonholonomic systems. In addition, we also generalize the Poincaré invariant for the conservative and nonconservative system with nonlinear nonholonomic constraints. Some example of the generalized Noether's theorem are given.

2. GENERALIZATION OF NOETHER'S THEOREM

Let the set of generalized coordinates of a conservative system be q_i $(i=1,2,\ldots,n)$. The Lagrangian of this system is $L(t, q_i, \dot{q}_i)$. The system is subjected to nonlinear nonholonomic constraints

$$G_s = G_s(t, q_i, \dot{q}_i) = 0 \qquad (s=1,2,\ldots,m, m<n) \tag{1}$$

Suppose that under the infinitesimal transformation

$$\begin{aligned} t \to t' &= t + \Delta t = t + \varepsilon_\sigma \tau^\sigma(t, q_i, \dot{q}_i) \\ q_i(t) \to q_i'(t') &= q_i(t) + \Delta q_i(t) = q_i(t) + \varepsilon_\sigma \xi_i^\sigma(t, q_i, \dot{q}_i) \end{aligned} \tag{2}$$

the change of the Lagrangian is a total time derivative of the form $\varepsilon_\sigma \Omega^\sigma(t, q_i, \dot{q}_i)$, where ε_σ $(\sigma=1,2,\ldots,k)$ are small parameters. (Note the summation of repeated up and down indices.) Since $\Delta q_i = \delta q_i + \dot{q}_i \Delta t$, it follows that

$$\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right](\xi_i^\sigma - q_i\tau^\sigma] + \frac{d}{dt}\left[L\tau^\sigma + \frac{\partial L}{\partial \dot{q}_i}(\xi_i^\sigma - \dot{q}_i\tau^\sigma) - \Omega^\sigma\right] = 0 \tag{3}$$

Under the transformation (2), from the change of the expressions (1) we get

$$\delta G_s = \frac{\partial G_s}{\partial \dot{q}_i}(\xi_i^\sigma - \dot{q}_i\tau^\sigma)\varepsilon_\sigma = K_s^\sigma \varepsilon_\sigma \tag{4}$$

Using the Lagrangian multiplier λ^s, combining the expressions (3) and (4), along the trajectory of the motion (Mei, 1985), one obtains

$$\frac{d}{dt}\left[L\tau^\sigma + \frac{\partial L}{\partial \dot{q}_i}(\xi_i^\sigma - \dot{q}_i\tau^\sigma) - \Omega^\sigma\right] = \lambda^s K_s \tag{5}$$

Suppose the generators τ^σ, ξ_i^σ of the transformation (2) satisfy the following conditions:

$$\frac{\partial G_s}{\partial \dot{q}_i}(\xi_i^\sigma - \dot{q}_i\tau^\sigma) = 0 \tag{6}$$

Then from the expressions (5) we have

$$L\tau^\sigma + \frac{\partial L}{\partial \dot{q}_i}(\xi_i^\sigma - \dot{q}_i\tau^\sigma) - \Omega^\sigma = \text{const} \tag{7}$$

According to the constraint of Chetaev type (Mei, 1985), expressions (6) imply that the constraints to impose the virtual displacements must satisfy those conditions. Hence, we have the following generalized Noether theorem. If the Lagrangian is invariant under the transformation (2) up to a total derivative and the generators τ^σ, ξ_i^σ satisfy the conditions (6), then there are some conservation laws (7) for the conservative nonlinear nonholonomic constrained system.

For a nonconservative and nonholonomic system of Chataev type, the Routh equations of motion are (Mei, 1985)

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)=Q^i+\lambda^s\frac{\partial G_s}{\partial \dot{q}_i} \tag{8}$$

where $L(t, q_i, \dot{q}_i)$ is a Lagrangian function which describes the conservative part of the system. $Q^i = Q^i(t, q_i, \dot{q}_i)$ are the components of the nonconservative generalized forces. Suppose that $\delta q_i = (\xi_i^\sigma - \dot{q}_i\tau^\sigma)\varepsilon_\sigma$ represents the virtual displacement. According to the D'Alembert–Lagrange principle, we have

$$\left[\frac{\partial L}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-Q^i\right](\xi_i^\sigma-\dot{q}_i\tau^\sigma)=0 \tag{9}$$

From (2) and (9) one obtains (Vujanovic, 1978)

$$\begin{aligned}\frac{d}{dt}&\left[L\tau^\sigma+\frac{\partial L}{\partial \dot{q}_i}(\xi_i^\sigma-\dot{q}_i\tau^\sigma)-\Omega^\sigma\right]\\&=\frac{\partial L}{\partial q_i}\xi_i^\sigma+\frac{\partial L}{\partial \dot{q}_i}(\dot{\xi}_i^\sigma-\dot{q}_i\dot{\tau}^\sigma)+\frac{\partial L}{\partial t}\tau^\sigma+L\dot{\tau}\\&\quad+Q^i(\xi_i^\sigma-\dot{q}_i\tau^\sigma)-\dot{\Omega}^\sigma\end{aligned} \tag{10}$$

If the generators τ^σ, ξ_i^σ of the transformation (2) satisfy conditions (6) and if the relations

$$\frac{\partial L}{\partial q_i}\xi_i^\sigma+\frac{\partial L}{\partial \dot{q}_i}(\dot{\xi}_i^\sigma-\dot{q}_i\dot{\tau}^\sigma)+\frac{\partial L}{\partial t}\tau^\sigma+L\dot{\tau}^\sigma+Q^i(\xi_i^\sigma-\dot{q}_i\tau^\sigma)-\dot{\Omega}^\sigma=0 \tag{11}$$

are satisfied, then there are some conservation laws (7) of this system. Since τ^σ and ξ_i^σ do not depend on $\ddot{q}_i$, (11) can be split into a system of linear partial differential equations with respect to τ^σ and ξ_i^σ, which is obtained by equating terms of corresponding degrees in $\ddot{q}$:

$$\begin{aligned}\tau^\sigma\frac{\partial L}{\partial t}+\xi_i^\sigma\frac{\partial L}{\partial q_i}+\left(\frac{\partial \xi_i^\sigma}{\partial t}+\frac{\partial \xi_i^\sigma}{\partial q_j}\dot{q}_j-\frac{\partial \tau^\sigma}{\partial t}\dot{q}_i-\frac{\partial \tau^\sigma}{\partial q_j}\dot{q}_i\dot{q}_j\right)\frac{\partial L}{\partial \dot{q}_i}+L\left(\frac{\partial \tau^\sigma}{\partial t}+\frac{\partial \tau^\sigma}{\partial q_i}\dot{q}_i\right)\\+Q^i(\xi_i^\sigma-\dot{q}_i\tau^\sigma)-\frac{\partial \Omega^\sigma}{\partial t}-\frac{\partial \Omega^\sigma}{\partial q_i}\dot{q}_i=0\end{aligned} \tag{12}$$

$$\frac{\partial L}{\partial \dot{q}_i}\left(\frac{\partial \xi_i^\sigma}{\partial \dot{q}_k}-\frac{\partial \tau^\sigma}{\partial \dot{q}_k}\dot{q}_i\right)+L\frac{\partial \tau^\sigma}{\partial \dot{q}_k}-\frac{\partial \Omega^\sigma}{\partial \dot{q}_k}=0 \tag{13}$$

We call this system of partial differential equations (6), (12), and (13) the generalized Killing equations. When L, Q^i, G_s are given and the solution τ^σ, ξ_i^σ, Ω^σ of equations (6) and (11) or of the generalized Killing equations are found, then conserved quantities of the form (7) automatically exist. Both results are a generalization of Noether's theorem for a nonconservative holonomic system (Vujanovic, 1978) or a linear nonholonomic system (Li, 1981, 1984; Bahar and Kwatny, 1987).

3. GENERALIZATION OF POINCARÉ INVARIANT

Suppose the Lagrangian of the conservative system is $L(t, q_i, \dot{q}_i)$, the system being subject to constraints (1). Consider the transformation

$$\begin{aligned} t &\to t' = t + \Delta t(\alpha) \\ q_i(t) &\to q_i'(t', \alpha) = q_i(t) + \Delta q_i(t, \alpha) \\ \dot{q}_i(t) &\to \dot{q}_i'(t', \alpha) = \dot{q}_i(t) + \Delta \dot{q}_i(t, \alpha) \end{aligned} \tag{14}$$

where α is a parameter which satisfies

$$q_i'(t, 0) = q_i(t), \qquad \dot{q}_i'(t, 0) = \dot{q}_i(t) \tag{15}$$

Combining the change of the action and expressions (4), one obtains

$$\begin{aligned} \delta I = I'(\alpha)\delta\alpha &= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \lambda^s \frac{\partial G_s}{\partial \dot{q}_i}\right] \delta q_i \, dt \\ &\quad + \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(L\,\Delta t + \frac{\partial L}{\partial \dot{q}_i}\,\delta q_i\right) + \lambda^s K_s^\sigma \varepsilon_\sigma\right] dt \end{aligned} \tag{16}$$

Along the trajectory of the motion

$$\delta I = I'(\alpha)\,\delta\alpha = \left(L\,\Delta t + \frac{\partial L}{\partial \dot{q}_i}\,\delta q_i\right)_1^2 + \int_{t_1}^{t_2} \lambda^s \frac{\partial G_s}{\partial \dot{q}_i}\,\delta q_i \, dt \tag{17}$$

If the simultaneous variation δq_i determined by the transformation (14) satisfies the same conditions as the virtual displacement imposed by constraints along the trajectory, it follows that

$$\delta I = I'(\alpha)\,\delta\alpha = \left[\left(L - \frac{\partial L}{\partial \dot{q}_i}\,\dot{q}_i\right)\Delta t + \frac{\partial L}{\partial \dot{q}_i}\,\Delta q_i\right]_1^2 \tag{18}$$

Suppose that in the space of the variables t, q_i, and $\dot{q}_i$ there is a closed curve C_1 which satisfies the constraint conditions (1), the equation of C_1 being given by

$$t = t^{(1)}(\alpha), \qquad q_i = q_i^{(1)}(\alpha), \qquad \dot{q}_i = \dot{q}_i^{(1)}(\alpha) \tag{19}$$

where $\alpha = 0$ and $\alpha = l$ are some points on C_1. Through any point on C_1 there is a trajectory of the motion; these trajectories through every point on C_1 form a tube of trajectories

$$q_i = q_i(t, \alpha), \qquad \dot{q}_i = \dot{q}_i(t, \alpha) \tag{20}$$

where $q_i(t, 0) = q_i(t, l)$, $\dot{q}_i(t, 0) = \dot{q}_i(t, l)$. Choose another closed curve C_2 on this tube of trajectories such that C_2 intersects the generatrix of the tube once. Suppose the equation of C_2 is given by

$$t = t^{(2)}(\alpha), \qquad q_i = q_i^{(2)}(\alpha), \qquad \dot{q}_i = \dot{q}_i^{(2)}(\alpha) \tag{21}$$

Along the curves C_1 and C_2 take the integral of the expression (18), which gives the same result respectively:

$$\oint_{C_i} \left[\left(L - \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \Delta t + \frac{\partial L}{\partial \dot{q}_i} \Delta q_i \right] = \text{inv} \tag{22}$$

That is to say, if δq_i determined by (14) just represents virtual displacements, then we can obtain the generalized Poincaré invariant (22) for a conservative nonlinear nonholonomic system of Chateav type.

For a nonconservative nonlinear nonholonomic system, under the transformation (14), consider the contemporaneous variational expression

$$\delta \int_{t_1}^{t_2} L\, dt - \int_{t_1}^{t_2} Q^i\, \delta q_i\, dt \tag{23}$$

Using a similar manipulation as was used to find expression (16), we can express (23) as

$$\int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) - \lambda^s \frac{\partial G_s}{\partial \dot{q}_i} - Q^i \right] \delta q_i\, dt$$
$$+ \int_{t_1}^{t_2} \left[\frac{d}{dt} \left(L\, \Delta t + \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) + \lambda^s K_s^\sigma \varepsilon_\sigma \right] dt \tag{24}$$

If δq_i satisfy the conditions as virtual displacements imposed by constraints, according to the equations of motion (8), the expression (24) is a total differential of α. Hence, we can also obtain the generalized Poincaré invariant (22) for a nonconservative nonlinear nonholonomic system of Chetaev type.

4. EXAMPLES

Consider a mechanical system whose Lagrangian is

$$L = T - V = \tfrac{1}{2} \sum_{i=1}^{4} \dot{x}_i^2 - \frac{1}{r^2}, \qquad r^2 = \sum_{i=1}^{4} x_i^2 \tag{25}$$

The system is subject to a constraint

$$G=G(t, x_i, \rho_1, \rho_2)=0 \tag{26}$$

where $\rho_1=x_2\dot{x}_1-x_1\dot{x}_2$, $\rho_2=x_4\dot{x}_3-x_3\dot{x}^4$, and G is a homogeneous function with respect to ρ_1 and ρ_2. It is easy to verify that $\tau^\sigma=t$, $\xi_i^\sigma=\frac{1}{2}x_i$, $\Omega^\sigma=0$ is a solution of equations (6) and (11). Thus, the expression (7) produces a conservation law

$$\frac{1}{2}\sum_{i=1}^{4} x^i\dot{x}_i-(T+V)t=\text{const} \tag{27}$$

Next, let the Lagrangian of the system be $L=T-V=\frac{1}{2}a^{ij}(q)\dot{q}_i\dot{q}_j-V(q)$, which describes the conservative part of the system, where V is the potential energy, and the dissipative forces are given by

$$Q^i=\frac{1}{2T}\frac{\partial T}{\partial \dot{q}_i}\frac{dR}{dt} \tag{28}$$

where R is a given function of time, generalized coordinates q_i, and generalized velocities $\dot{q}_i$. Suppose the constraint equations are homogeneous with respect to $\dot{q}_i$. Obviously, $\tau^\sigma=0$, $\xi_i^\sigma=\dot{q}_i$ satisfy equation (6), and equation (11) becomes

$$\frac{\partial L}{\partial q_i}\dot{q}_i+\frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i+\frac{dR}{dt}-\dot{\Omega}=0 \tag{29}$$

Hence,

$$\Omega=L+R \tag{30}$$

From the expression (7), it follows that

$$T+V-R=\text{const} \tag{31}$$

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